Determination of the gravitational lapse rate

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The fundamental dynamic process in the energy dynamics of the atmosphere is the creation of the lapse rate – the rate that the temperature drops with increasing altitude in the troposphere – below the tropopause marked by a dotted line in Figure 1 where the Earth curve follows a straight line. The tropopause is not a fixed height. It can vary from close to zero altitude at the poles to over 20 km at the equator. It varies in time, and thunderstorms can push it up locally. A typical height is said to be 11 km.

Some people think that the lapse rate is entirely due to radiative gasses (aka greenhouse gasses) and without them the atmosphere would have a constant temperature all the way up – be isothermal. The Postmodern view of the radiative dynamics of the atmosphere is based on this assertion.

It is a plausible first assumption, since we know that hot air rises. We might even expect to have cold air at the bottom and hot at the top, except that the atmosphere is mainly heated from the bottom. The problem is that these views are based on thermodynamics for laboratory conditions, which generally ignores gravity because the effect of gravity over small height changes is negligible.

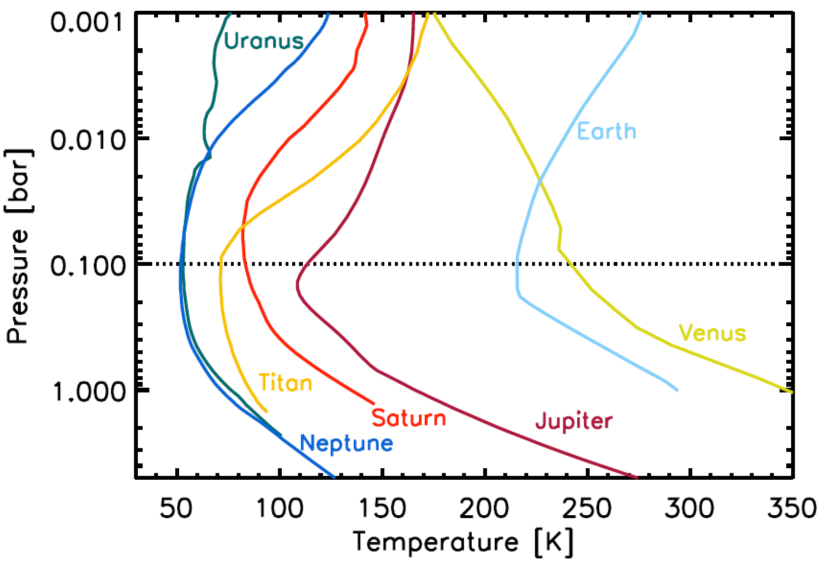

Figure 1: Atmospheric temperatures (1)

There are several definitions of lapse rate and some confusion in their use, so I'll start by giving definitions as I prefer to use them:

- Dry adiabatic lapse rate, DALR or ALR: with no radiative gasses
- Gravitational lapse rate, GLR: my preferred name for ALR
- Moist lapse rate, MLR: air with moisture levels below saturation
- Saturated lapse rate, SLR: air with water vapour at saturation levels
- Environmental lapse rate, ELR: an actual lapse rate at a particular place and time

The ALR is usually calculated from the thermodynamics of a parcel of air rising up through the troposphere. Air can't be adiabatic. Adiabatic means no energy is lost or gained by the gas parcel, which excludes radiative gasses which would transfer infrared energy in and out of the parcel, so the ‘dry’ is superfluous. The ALR applies only to an idealised mixture of gasses such as nitrogen and oxygen that are not radiative at atmospheric temperatures, so it is a theoretical abstraction. It provides the foundation of the actual lapse rate, which is modified by the addition of RGs. Thermodynamics gives a formula for calculating the lapse rate:

$$\Gamma_{th} = g/c_p \qquad \text{(E1)}$$

Where g is the gravitational acceleration and c_p is the specific heat of air at constant pressure – a measure of the amount of energy needed to raise the temperature of the gas.

I find the derivation of this formula too opaque. It hides the basic physics, which has caused a great deal of confusion and controversy (note b). After being resolved over a century ago, the issue has surfaced again in recent years in an effort to exaggerate the role of radiative gasses.

In this essay I go down to the level of individual molecules and give an alternative derivation for the ALR. The basic physics is simple. If you throw a ball into the air its energy can be given as the sum of its energy of movement – its kinetic energy, E_K – and its gravitational potential energy, E_P, minus energy lost to friction with the air, which can be ignored if the ball is in a vacuum. It moves up until all its energy is potential energy, then starts to fall and regain it.

$$E = E_K + E_P = mv^2/2 + mgh \qquad \text{(E2)}$$

Where m is the mass of the ball, v is its velocity or speed, h is its height, and g is the gravitational acceleration.

An insight into the lapse rate problem can be gained from the fact that a ball falling in a vacuum from a hight of 11 km has a velocity at ground level of 464 m/s, which is precisely the mean velocity of air molecules at 20 C° (2), and 11 km is a typical hight of the tropopause. This, and the suggestive g in E1, was the starting point that prompted me to try the following analysis.

Between collisions, the molecules that constitute air behave just like the ball. Having a molecule falling in a vacuum may not seem relevant when we're considering the atmosphere, but between collisions with other molecules they actually are all falling in a vacuum, or close enough for a simple analysis. All the molecules are following a parabolic path and gaining a little downward energy between collisions. Those moving down will gain kinetic energy, and those moving up will lose it. This produces a gradient with the average kinetic energy of molecules decreasing with increasing altitude – in other words, a temperature gradient.

Eventually our falling molecule will hit other ones, and the energy it has gained in falling will be passed on to them. The gravitational energy will be thermalised – added to the random motion of other molecules, to their kinetic energy, until an equilibrium is established.

If you want to skip the detail, go to E5. The next step is the most technical one because we aren't dealing with billiard balls colliding. We have to divide the added energy among all the degrees of freedom of the molecules, f. This is the standard equipartition rule dictated by entropy – the energy will distribute between all possible modes for storing it. Nitrogen and oxygen have 5 degrees of freedom at atmospheric temperatures. That's 3 for the directions of motion and 2 for rotational motions – spin and tumbling – rotation around the axis joining the two atoms doesn't count. I'll call this f_m.

During a collision we also have to consider the resulting energy distribution between the two colliding molecules. That's four nuclei and around 30 electrons. This needs to be seen from a quantum mechanical perspective as the transient formation of a four atom molecule which passes – slowly by atomic standards – through a sequence of vibrational and rotational quantum states as it tries to form then breaks up – a process which determines the final distribution of energy between the molecules, and gives 2 more degrees of freedom, f_c.

The temperature of a gas is related to its kinetic energy by:

$$E_K = fkT/2 \qquad \text{(E3)}$$

where T is the temperature and k is Boltzmann's constant. The energy gained by a molecule falling a distance Δh is $mg\Delta h$ (the deltas indicating related changes) so partitioning this among the available degrees of freedom we get:

$$\Delta E = mg\Delta h = fk\Delta T/2 \qquad \text{(E4)}$$

After a little manipulation we have the temperature gradient or lapse rate as:

$$\Gamma_g = \Delta T/\Delta h = 2mg/(f_m + f_c)k \qquad \text{(E5)}$$

Plugging in some numbers, m is taken as the average mass of nitrogen and oxygen in air weighted by their relative proportions of 79:21.

Comparing the two approaches, using a value for g adjusted slightly for a mean troposphere altitude of 5.5 km reduces it by about 0.8% from the usual surface value, and c_p is taken as the measured value of 1.0035.

E1 gives 9.73 C°/km.

E5 gives 9.66 C°/km.

Within 1% difference they are close, given that the real world doesn't usually comply exactly with simple physical theory.

Next, I demonstrate the theoretical equivalence of the gravitational lapse rate and the conventional derivation, so if the theoretical value for c_p is used in E1 the two approaches give exactly the same result.

Equating Γ_{th} with Γ_g – for those who are comfortable with some simple algebra and cryptic physics. I'm assuming a perfect monomolecular gas.

Starting with E5: $\Gamma_g = 2mg/(f_m + f_c)k$.

Substituting M/N_A for m, where M is the molecular mass of the molecule and N_A is Avagadro's number gives:

$$\Gamma_g = 2Mg/N_A(f_m + f_c)k \qquad \text{E6}$$

Substituting R/N_A for k, where R is the gas constant, gives:

$$\Gamma_g = 2gM/(f_m + f_c)R \qquad \text{E7}$$

Now looking at the conventional derivation in E1: $\Gamma_{th} = g/c_p$.

We can derive a theoretical value for c_v starting with c_{vm}, the molecular heat capacity:

Combining $c_{vm} = f_m R/2$, $c_{pm} = R + c_{vm}$, and $c_p = c_{pm}/M$ gives:

$$c_p = (f_m + 2)R/2M \qquad \text{E8}$$

Substituting this in E1 we get:

$$\Gamma_{th} = 2gM/(f_m + 2)R \qquad \text{E9}$$

So with $f_c = 2$ in E7 we have:

$$\Gamma_g = \Gamma_{th} \qquad \text{E10}$$

The two derivations for the adiabatic lapse rate are theoretically equivalent. That these two distinct approaches can be shown to reduce to the same dependence on g provides strong confirmation of the role of gravity in the base lapse rate.

The addition of radiative gasses produces, effectively instantaneous, energy transfer by radiation over average distances of tens of metres at ground level, increasing to kilometres through to infinity – to outer space – at the tropopause as the air gets thinner.

References:

1. Robinson T.D., Catling, D.C., *Common 0.1 bar tropopause in thick atmospheres set by pressure-dependent infrared transparency*, Nature Geoscience Letters, 8 December 2013
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